NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1251

ON THE FORMATION OF SHOCK WAVES IN SUBSONIC FLOWS WITH LOCAL SUPERSONIC VELOCITIES

By F. I. Frankl

Translation

"K Obrazovaniu Skachkov Uplotnenia v Dozvukovykh Techeniakh s Mestnymi Sverkhzvukovymi Skorostiami." Prikladnaya Matematika i Mekhanika, Vol. XI, 1947.

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In the flow about a body with large subsonic velocity if the velocity of the approaching flow is sufficiently large, regions of local supersonic velocities are formed about the body. It is known from experiment that these regions downstream of the flow are always bounded by shock waves; a continuous transition of the supersonic velocity to the subsonic under the conditions indicated has never been observed.

A similar phenomenon occurs in pipes. If at two cross sections of the pipe the velocity is subsonic and between these sections regions of local supersonic velocity are formed without completely occupying a single cross section, these regions are always bounded by shock waves.

A theoretical explanation of this phenomeon has as yet not been given. In 1932, the author (reference 1) constructed an example of a plane parallel continous (that is, without shocks) adiabatic flow in a channel containing a local supersonic region in contact with one of the walls. The attempt at the experimental realization of this flow gave, however, a negative result (reference 2). Since that time, many theoretical examples have been constructed by various authors but not one of them has been realized experimentally.

The opinion has also been expressed that small supersonic regions without shock waves were possible but for too large dimensions a continuous flow becomes impossible in view of the superposing on each other of Mach lines of the same family. This assumption was, however, refuted in the recent work of A. Nikolskii and Taganov (reference 3). (See equation (2.26) and theorem 8 in reference 3.)

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*"K Obrazovaniu Skachkov Uplotnenia v Dozvukovykh Techeniakh s Mestnymi Sverkhzvukovymi Skorostiami." Prikladnaya Matematika i Mekhanika, Vol. XI, 1947, pp. 190-202.

It is shown herein that the problem of finding a continuous flow about an arbitrary contour with local supersonic velocities is not legitimately posed and, generally speaking, has no solution.[1] The conclusions herein are based on a single hypothesis having the character of a uniqueness theorem. (See footnote 1.)

Although a contour may be given, which for a given velocity at infinity may have a flow about it with supersonic velocities, other contours exist very close to it (not only as regards the coordinates of the corresponding points but also the slopes, curvatures, and so forth) for which the problem has no solution. It therefore follows that if, in general, there may at all be found a steady flow about an arbitrary given contour with local supersonic velocities then this flow must be accompanied by shock waves.

This assumption is now considered in greater detail. Only continuous adiabatic irrotational study flows with constant entropy and constant total energy are considered. The discussion is restricted to the case of flow about bodies inasmuch as flows in pipes can be analogously considered.

The considerations are greatly simplified if it is assumed that the profile is an oval with two axes of symmetry, these axes being taken as the axes of coordinates. It is assumed that the velocity of flow at infinity is less than the velocity of sound and parallel to the x-axis and also that the x-axis is a streamline,

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[1]As is correctly remarked by Nikolskii and Taganov, the existing attempts to solve this problem are based on processes of successive approximations, the convergence of which have not been shown. (See, for example, reference 6.) Taylor, who likewise tried to solve this problem by successive approximations, showed the practical convergence of the process in the presence of supersonic velocities (reference 7). Hence, from these works there does not follow the existence of a solution and the attempts to make use of these doubtful approximate solutions for deriving a certain critical Mach number below which the problem is solved (without shocks) is unfounded.

That the problem is not solved for arbitrary contours follows also from theorem 6 of Nikolskii and Taganov. This theorem states that subsonic flows without shocks with supersonic regions about a body consisting of straight segments are impossible. Even in the presence of the very smallest straight segments of the contour at the supersonic region, flow without shocks is impossible.

1256

(in other words, the problem is restricted to the case of the flow about a contour without circulation).

It is now assumed as a hypothesis that such flows must be symmetrical not only with respect to the x-axis but also with respect to the y-axis. In fact, examples contradicting this assertion are not known. Flows that are nonsymmetrical with respect to the y-axis are known only in the case of the presence of shock waves.

A flow of the kind considered about the contour S that has at infinity the velocity U is now assumed.

Let the regions of supersonic velocities be bounded by the arcs ABC and A'B'C' (fig. 1). Let BD and BE be Mach lines ending at the point B.

Because the flows in all the four quadrants formed by the x- and y-axes are the same, only the flow in the quadrant $x < 0$ and $y > 0$ are considered. The boundary conditions for the stream function are

(1) $\psi = 0$ on the segment $-\infty F$ of the x-axis
(2) $\psi = 0$ on the arc FH of the contour S
(3) $\partial\psi/\partial x = 0$ on the segment $H\infty$ of the y-axis
(4) The velocity at infinity is equal to U.

In order to realize these boundary conditions, the following mechanical model is applied. Let the segment $-\infty F$ of the x-axis and the arc FH of the contour be taken as rigid walls; on the segment $H\infty$ of the y-axis, an infinite chain of infinitely closely distributed pumps controlled by an automatic regulator are set up. This regulator is to maintain the velocity at infinity at a given level and moreover is not to permit the formation of flows on the y-axis that are parallel to this axis (condition 3).

If the shape of the wall at the segment DH is changed, it is evident that the effect of this change cannot be propagated to the left of the Mach line DB. In the same way in this region to the left of the Mach line DB, the effect of a change in the working regime of the pumps cannot be propagated to the segment HB provided they continue to maintain the velocity at the segment. Hence, the flow in the part of the field lying to the left of the Mach line DB is uniquely determined on the basis of the boundary conditions

(1') $\psi = 0$ on the segment $-\infty F$ of the x-axis
(2') $\psi = 0$ on the arc FD of the contour S
(3') $\partial\psi/\partial x = 0$ on the segment $B\infty$ of the y-axis
(4') The velocity at infinity is equal to U.

On the basis of symmetry considerations, this result is carried over to the remaining three quadrants.

Hence, the shape of the arcs D'FD and E'GE of the contour S completely determines the flow outside the closed curve FDBEGE'B'D'F formed from the two arcs of the contour S and the four Mach lines.

Inasmuch as the distributions of the velocities on the Mach lines BD and BE in the curved quadrangle DBEKD are known where DK and EK are the remaining two Mach lines passing through D and E, the velocity field is entirely determined.

The arc DE of the contour, however, cannot be chosen independently of the shapes of the arcs D'FD and E'GE. The smallest change in it makes the solution of the given boundary problem impossible.

Under actual conditions there always exist, of course, small disturbances. Hence, the flow about the contour of the type considered of a continuous (without shocks) subsonic flow with local supersonic velocities is in general impossible.

Although the posing of the problem of the continuous flow about a prescribed contour in the presence of supersonic regions is considered invalid, it is probable that for sufficiently wide conditions the existence and uniqueness theorems of the continous flow hold for a profile partly given in this sense.

An exact formulation of the existence and uniqueness theorems proposed is now given, removing the restriction that the flow about the contour is symmetrical about the y-axis (fig. 2).

In the x,y-plane, let the contour S be given, which is symmetrical about the x-axis.

All adabiatic continous steady flows of a gas satisfying the conditions are considered:

(a) At infinity, all the particles of the gas have the same state parameters and velocity U parallel to the x-axis.

1256

(b) The flow is symmetrical with respect to the x-axis.

Then either one of the following possibilities holds:

(A) There exists one and only one flow determined everywhere outside the contour S and satisfying conditions (a) and (b).

(B) If there is no such flow, then there exists a flow determined outside the closed curve FDBEGE'B'D'F, where D'FD and E'GE are arcs of the contour S and BD and BE, B'D', and B'E' are pairs of Mach lines symmetrical with respect to the x-axis; the velocity vector at the point B must form with the x-axis an arbitrarily (within certain limits) given angle θ. The flow must, of course, satisfy the conditions (a) and (b).

On the contour S it is necessary to impose certain smoothness conditions, which must be further specified.

In case (B), the flow continues (with the aid of the solution of the problem of Goursat) uniquely in the region included between the Mach lines BD and BE and correspondingly between B'D' and B'E'. The flow thus passes a certain contour S coinciding with the contour S^* along the arcs D'FD and E'GE.

The arcs DE and D'E' of the curve S^* are uniquely determined and, in general, differ from the corresponding arcs of the contour S.

If this preceding formulated existence and uniqueness theorem (or at least uniquenes) is proved, then in particular there is proven the nonvalidity of the statement of the problem of continuous adiabatic steady flow with local supersonic regions about a body.

If the boundary problem (B) is solved for a certain contour S^*, then for the neighboring contour S it may easily be mapped on the plane of the hodograph (fig. 3).

Let η be the velocity function introduced in reference 4.

Let the contour S^* correspond to the stream function ψ^* and the contour S to the stream function $\psi = \psi^* + \delta\psi$. The functions ψ^*, ψ, and $\delta\psi$ satisfy the equation

$$\eta \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\eta^2} + b(\eta) \frac{\partial\psi}{\partial\eta} = 0$$

The lines EG and DF on figure 3 correspond to the arcs EG and DF of the contour S*, the corresponding points having the same notation. The lines BE and BD are characteristics. The points G and F on figure 3 are infinitely removed (the coordinates are $-\pi/2$, $+\infty$, and $+\pi/2$, $+\infty$, respectively). The point η_0 on the η-axis corresponds to the velocity U of the flow at infinity. The boundary conditions for $\delta\psi$ will then be the following:

1. On the lines GCE and FAD

$$M\delta\psi + N\frac{\partial\delta\psi}{\partial\theta} + P\frac{\partial\delta\psi}{\partial\eta} = f(\eta)$$

where M, N, and P are functions of η depending on the contour S* and $f(\eta)$ is a function depending on the difference between the contours S and S*.

2. On both sides of the sequent $(\eta_0 + \infty)$ along the η-axis, $\delta\psi = 0$.

3. On approaching the point $(0, \eta_0)$, the function $\delta\psi$ approaches infinity.

4. As $\eta \to \infty$, the magnitude $\delta\psi$ approaches sufficiently rapidly to zero.

The close connection of this problem with the problem of Tricomi (reference 5) and other already solved problems for the equations of the mixed elliptico-hyperbolic type speaks in favor of the existence and uniqueness of its solution.

Translated by S. Reiss,
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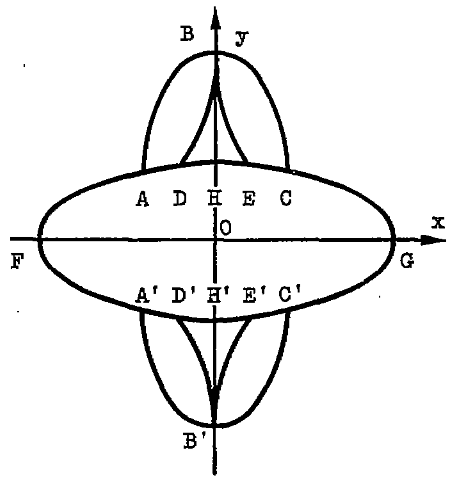

Figure 1.

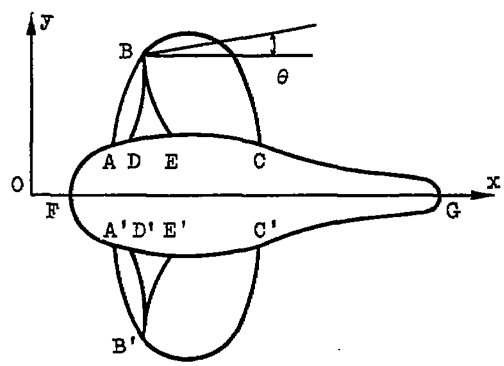

Figure 2.

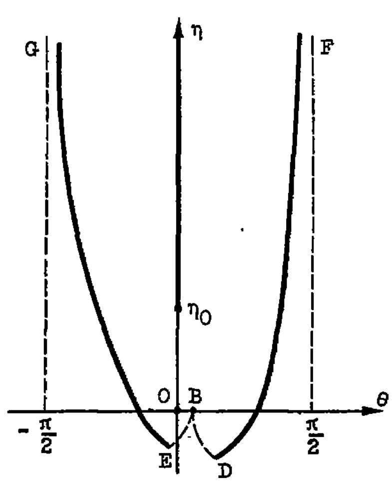

Figure 3.